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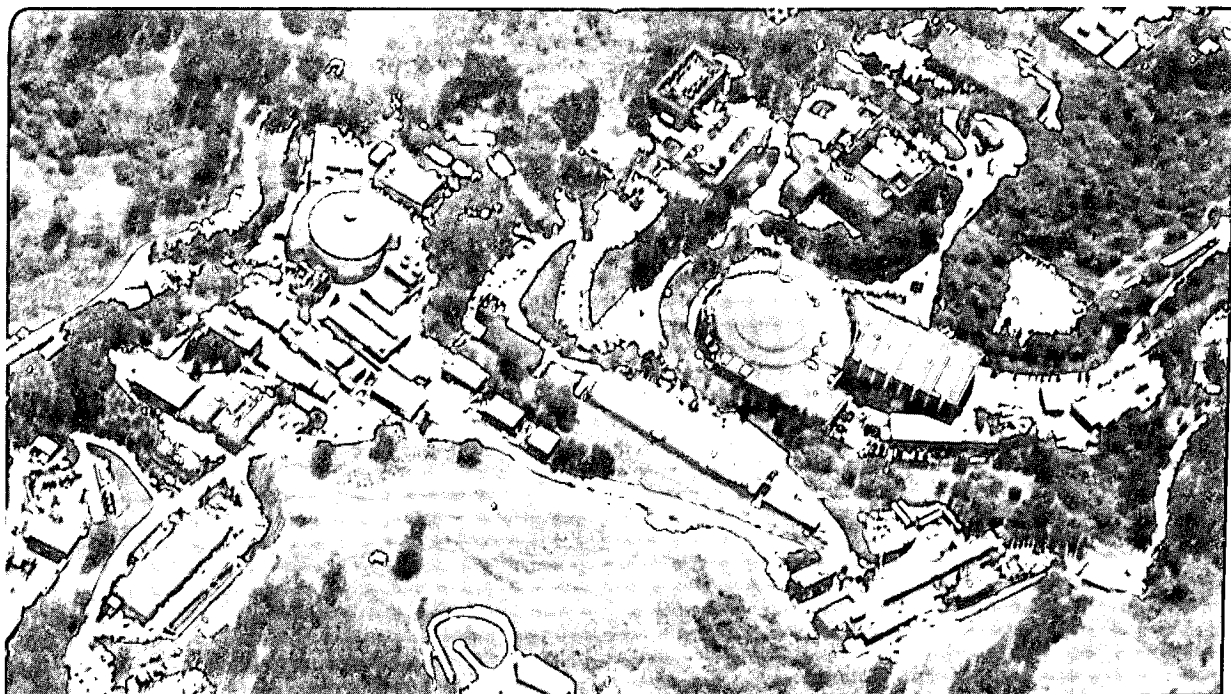
Mathematics Department

To be submitted for publication

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June 1992



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TURBULENCE CALCULATIONS IN MAGNETIZATION VARIABLES

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June 1992

¹ Partially supported by the Applied Mathematical Sciences Subprogram of the Office of Energy Research, U.S. Department of Energy under Contract DE-FG02-88ER25053 and the National Science Foundation under Grant DMS-9100383.

² Partially supported by the Applied Mathematical Sciences Subprogram of the Office of Energy Research, U.S. Department of Energy under Contract DE-AC03-76SF00098.

Abstract

The equations of incompressible fluid dynamics in three dimensions are reformulated in terms of magnetization variables; the usefulness of the resulting equations in turbulence theory and in computational fluid dynamics is explained. In particular, the new variables provide a promising avenue to real-space renormalization in fluid dynamics.

DEDICATED TO SAUL ABARBANEL, ON THE OCCASION OF HIS 60TH BIRTHDAY.

Introduction. In the case of two-dimensional flow, the motion of vortices and vortex blobs is described by a simple Hamiltonian structure that is the starting point for numerical algorithms (see e.g. [3],[13]) as well as for statistical mechanical analyses of two-dimensional turbulence (see e.g. [11],[15],[16],[20],[22],[26]). In principle, this structure could be the starting point for renormalization procedures for simplifying numerical calculations; it has not been used in this way since in two dimensions the energy cascade runs towards the large scales that one does not want to renormalize away.

Euler's equations form a Hamiltonian system in three dimensions as well as in two. It turns out [1] that one of the Hamiltonian structures that have been proposed (see e.g. [23],[27],[28]) leads to convergent discretizations that are both simple and Hamiltonian at every level of discretization and that can be used to construct three-dimensional analogues of various two-dimensional numerical and statistical arguments [1]. In the framework provided by this "magnetization" representation renormalization appears in a particularly simple guise. Our goal here is to summarize this development.

The two-dimensional case. We begin by summarizing well-known two-dimensional constructions that will motivate some of the three-dimensional arguments. We assume the reader is familiar with the Euler and Navier-Stokes equations for the velocity $\underline{u} = (u_1, u_2)$ and the vorticity ξ . Space variables will be denoted by $\underline{x} = (x_1, x_2)$, t will be the time, and R the Reynolds number. We shall consider here only incompressible flow, $\text{div } \underline{u} = 0$.

Write

$$\xi \cong \sum_{i=1}^N \bar{\xi}_i \phi_\delta(\underline{x} - \underline{x}_i),$$

where $\phi_\delta = \delta^{-2} \phi(\underline{x}/\delta)$, $\int \phi = 1$.

Impart to the \underline{x}_i (“blob centers”) the motion

$$\frac{d\underline{x}_i}{dt} = \underline{u}(\underline{x}_i) = \sum_{\substack{i=1 \\ i \neq j}}^N K_\delta(\underline{x} - \underline{x}_j) \bar{\xi}_j, \quad \bar{\xi}_j = \text{constants}, \quad (1)$$

where $K_\delta = K * \phi_\delta$, $*$ = convolution, $K = (2\pi)^{-1}(-\partial_y, \partial_x) \log |\underline{x}|$. This motion approximates Euler flow (see e.g. [3],[13],[19],[24]). The system (1) is Hamiltonian, with Hamiltonian equal, after a scale change, to the stream function ψ . Its form is simplest in the case of point vortices, i.e, when ϕ_δ is a delta function:

$$H = -\frac{1}{4\pi} \sum_i \sum_{j \neq i} \bar{\xi}_i \bar{\xi}_j \log |\underline{x}_i - \underline{x}_j|. \quad (2)$$

To construct viscous flow, one only has to replace equation (1) by

$$d\underline{x}_i = \underline{u} dt + \sqrt{2/R} d\underline{w}(t) \quad (3)$$

where \underline{w} is two-dimensional Brownian motion ([3],[18]).

The same Hamiltonian is the starting point of Onsager’s statistical theory [22] and its generalizations (see e.g. [11],[15],[16],[20],[26]). In these theories, one defines a vortex temperature T that can be positive or negative; the transition of $|T| = \infty$ between positive and negative values of T resembles qualitatively the Kosterlitz-Thouless order/disorder transition [12],[17]. Order corresponds to $T < 0$. For $T < 0$, vorticity of one sign collects into coherent structures. The entropy is maximum when $|T| = \infty$.

Vortex generalizations to three dimensions. The constructions of the preceding section can be generalized to three space dimensions in several ways, replacing the “blobs” of two-dimensional vortex theory by (not necessarily divergence-free) arrows, or segments, or filaments of vorticity (for reviews, see e.g. [3],[13],[19],[24]). A statistical theory built on a filament representation has been presented in [5],[6],[8],[9]. Here too the vortex temperature T can be positive or negative. The maximum entropy state is at $|T| = \infty$ and is attracting. It separates order at $T < 0$

from disorder at $T > 0$; the positive temperature states are inaccessible for inviscid flows. Turbulence lives at $|T| = \infty$; a connection has been made [9] with the Shenoy-Williams three-dimensional analogue [30],[31] of the Kosterlitz-Thouless renormalization theory (see more below). This theory has been applied to the renormalization of numerical vortex algorithms [4],[7],[29].

It is apparent however that a vortex representation standing alone is not the optimal framework for applying statistical tools to numerical calculations (nor indeed, for extending the two-dimensional Kosterlitz-Thouless theory to three dimensions). The difficulty has to do with “localization”. Three-dimensional arrows or segments are only approximately solenoidal, and this allows numerical errors to grow and change asymptotic equilibria [1]. Vortex filaments can be made solenoidal, but then they are not “local”—they have one macroscopic dimension—and this leads to problems with boundaries, with the analogues of the random vortex equation (3), and with merger/renormalization procedures. The “magnets” we shall introduce in the next sections can be viewed as a device for “localizing” vortex filaments or vortex loops.

A gauge-invariant form of the Euler and Navier-Stokes equations. Consider the Euler equations in the form

$$\frac{D\underline{u}}{Dt} = -\text{grad}p, \quad \text{div } \underline{u}_t = 0. \quad (4)$$

In any domain Ω , with smooth enough boundary $\partial\Omega$, any smooth vector function \underline{w} can be written as a sum (the Hodge decomposition)

$$\underline{w} = \underline{u} + \text{grad}\phi$$

where $\text{div } \underline{u} = 0$, $\underline{u} \cdot \underline{n} = 0$ on $\partial\Omega$ (\underline{n} is the normal to $\partial\Omega$); $(\underline{u}, \text{grad}\phi) \equiv \int_{\Omega} \underline{u} \cdot \text{grad}\phi \, d\underline{x} = 0$, and thus \underline{u} is the orthogonal projection $P\underline{w}$ of \underline{w} on the space of solenoidal vectors parallel to $\partial\Omega$. If $\partial\Omega$ is at infinity, the boundary condition $\underline{u} \cdot \underline{n} = 0$ is replaced by a growth condition. $P\underline{u} = \underline{u}$,

$P\underline{u}_t = \underline{u}_t$, for solutions \underline{u} of (4), $P\text{grad}p = 0$; thus equation (4) can be written as

$$P(\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u}) = 0$$

or

$$\underline{u}_t + P((\underline{u} \cdot \nabla)\underline{u}) = 0 \quad (5)$$

The pressure p can always be recovered from

$$\text{grad}p = -(I - P)((\underline{u} \cdot \nabla)\underline{u}).$$

The definition $\underline{\xi} = \text{curl } \underline{u}$ shows \underline{u} to be the vector potential of the vorticity $\underline{\xi}$;

$\text{div } \underline{u} = 0$ is then the choice of gauge. Write

$$\underline{m} = \underline{u} + \text{grad}\phi$$

(it is not required that $\text{div } \underline{m} = 0$). Neither ϕ nor \underline{m} is unique; \underline{m} is a “magnetization”, the name will be explained below. Clearly, $\underline{u} = P\underline{m}$ for any choice of ϕ . One can readily verify that the system

$$\frac{Dm_i}{Dt} = -m_j \partial_i u_j, \quad \underline{m} = (m_1, m_2, m_3), \quad \underline{u} = P\underline{m}, \quad (6)$$

$\underline{m}(\underline{x}, 0)$ given, is equivalent to (5) for any initial choice of ϕ , and is the gauge-invariant form of Euler’s equations. The discussion of boundary conditions for (6) is omitted here, see for example [25]. Note further that the addition of a term $R^{-1} \Delta m_i$ to the first equation (6) produces a system equivalent to the projection form of the Navier-Stokes equations. The non-uniqueness of ϕ will be given a physical interpretation below.

As is well known, there is an analogy between fluid mechanics and magnetostatics, in which $\underline{\xi}$ corresponds to the current and \underline{u} corresponds to the magnetic induction; \underline{m} corresponds to the magnetization [14], hence the name and the notation.

Suppose $\underline{\xi} = \text{curl } \underline{u} = \text{curl } \underline{m}$, the vorticity, vanishes outside a sphere S of radius r . Since the exterior of S is simply connected in three dimensions, there exists a potential ϕ such that $\underline{u} = -\text{grad}\phi$ outside S . With this choice of ϕ , \underline{m} has support inside S . Thus, if $\underline{\xi}$ has compact support, one can construct \underline{m} with compact support. If \underline{m} has compact support, one can readily check that the impulse

$$\underline{I} = \int \underline{x} \times \underline{\xi} d\underline{x}$$

can be written as

$$\underline{I} = \int \underline{m} d\underline{x};$$

\underline{m} is also an impulse density, and is a conserved quantity in incompressible flow.

A discrete Hamiltonian approximation of Euler's equations. The vorticity $\underline{\xi}$ can always be written as a sum of divergence-free functions as small support—think of small vortex loops and see the explicit construction below. For each element of this sum, construct the magnetization whose support vanishes outside the support of the element. The resulting representation of the flow field is local and divergence-free.

More specifically, let $\phi_\delta = \delta^{-3} \phi(\underline{x}/\delta)$, $\int \phi = 1$, be a function whose support is $O(\delta^3)$; let ψ_δ satisfy $\Delta\psi_\delta = \phi_\delta$, $\Delta = \text{Laplace operator}$. Let $\underline{m}(\underline{x}) = \sum_{i=1}^N \underline{m}^{(j)} \phi_\delta(\underline{x} - \underline{x}_j)$, $\underline{m}^{(j)} = (\overline{m}_1^{(j)}, \overline{m}_2^{(j)}, \overline{m}_3^{(j)})$. The equation $\underline{u} = P\underline{m}$ yields a velocity field \underline{u} due to the magnetization in a blob ϕ_δ centered at \underline{x}_j :

$$\begin{aligned} \underline{u}^{(j)}(\underline{x}) &= \left(u_1^{(j)}(\underline{x}), u_2^{(j)}(\underline{x}), u_3^{(j)}(\underline{x}) \right), \\ u_\ell^{(j)}(\underline{x}) &= \overline{m}_\ell^{(j)} \phi_\delta(\underline{x} - \underline{x}_j) - \overline{m}_k^{(j)} \partial_k \partial_\ell \psi_\delta(\underline{x} - \underline{x}_j). \end{aligned}$$

The equation of motion of the ℓ -th “magnet” $\underline{m}^{(j)} \phi_\delta(\underline{x} - \underline{x}_j)$ is

$$\frac{d\underline{x}_\ell}{dt} = \underline{u}(\underline{x}_\ell) = \sum_{j=1}^N \underline{u}^{(j)}(\underline{x}_\ell), \quad (7)$$

with the evolution of the coefficients $\underline{m}^{(j)}$ given from equation (6) by

$$\frac{d\overline{m}_\ell^{(j)}}{dt} = -\overline{m}_k^{(j)} \partial_\ell u_k, \quad u_k \text{ the components of } \underline{u}(\underline{x}_\ell). \quad (8)$$

One can readily verify that the system (7)–(8) is Hamiltonian, with $H = \frac{1}{2} \sum_{j=1}^N \overline{\mathbf{m}}^{(j)} \cdot \underline{\mathbf{u}}(\underline{\mathbf{x}}_j)$. If the points $\underline{\mathbf{x}}_j$ are distributed so that the sum approximates an integral, then $H \sim \frac{1}{2} \int \underline{\mathbf{u}} \cdot \underline{\mathbf{m}} d\mathbf{x} = \frac{1}{2} \int \underline{\mathbf{u}} \cdot \underline{\mathbf{u}} d\mathbf{x}$ = the kinetic energy of the flow. One can readily verify that (7) is nothing but

$$\frac{d(\underline{\mathbf{x}}_\ell)_k}{dt} = \frac{\partial H}{\partial \overline{\mathbf{m}}_k^{(\ell)}}, \quad (\underline{\mathbf{x}}_\ell)_k \text{ the } k\text{-th component of } \underline{\mathbf{x}}_\ell, \quad (7')$$

and (8) is

$$\frac{d\overline{\mathbf{m}}_\ell^{(j)}}{dt} = -\frac{\partial H}{\partial (\underline{\mathbf{x}}_j)_\ell}. \quad (8')$$

Note that $\sum_{k=1}^N \overline{\mathbf{m}}^{(k)}$ and $\sum \underline{\mathbf{x}}_k \times \overline{\mathbf{m}}^{(k)}$ are vector constants of the motion for this system. If the $\underline{\mathbf{x}}_j$ are distributed in an irregular fashion H does not have to approximate the kinetic energy nor indeed be positive (for details, see [1]).

A comparison of these formulas with the formulas for the magnetic induction due to a current loop [14] show that the elements in the description just given can be viewed as small vortex loops. The velocity of a vortex loop is not uniquely defined by its first integrals but depends on the distribution of $\underline{\xi}$ in the loop. Correspondingly, the transformation

$$\underline{\mathbf{u}}(\underline{\mathbf{x}}_k) \rightarrow C_k \overline{\mathbf{m}}^{(k)} + \underline{\mathbf{u}}(\underline{\mathbf{x}}_k),$$

with C_k arbitrary constants, leaves the system Hamiltonian, with

$$H = \frac{1}{2} \sum_k \overline{\mathbf{m}}^{(k)} \cdot \left(\frac{1}{2} C_k \overline{\mathbf{m}}^{(k)} + \underline{\mathbf{u}}(\underline{\mathbf{x}}_k) \right).$$

The convergence of the discrete representation of this section, with appropriate choices for ϕ_δ , has been demonstrated in [1].

Vortex statistics in magnetization variables. We now consider how the properties of vortex systems are reflected in the properties of systems of magnets.

Vortex stretching should cause the sum

$$Z = \sum_{k=1}^N |\underline{m}^{(k)}|$$

to grow in time, indeed, form an entropy or Liapounov function. This can indeed be readily checked on the computer; a proof cannot be given at this time. It is particularly easy, in magnetization variables, to show that as vortex lines stretch they must fold; indeed, if $\sum |\underline{m}^{(k)}|$ increases while $\sum \underline{m}^{(k)}$ remains constant, cancellation must occur, i.e., vortex lines fold.

Consider now how to change variables from magnetization to vorticity and vice versa. Consider a macroscopic vortex loop C of circulation Γ . To construct a magnetization representation of C , construct a surface S that spans C . Its non-uniqueness corresponds to the non-uniqueness of \underline{m} and indeed explains it. Construct a coordinate system on S in terms of some variables, say q and s . In each rectangle \mathcal{R} with vertices (q, s) , $(q + \delta q, s)$, $(q + \delta q, s + \delta s)$, $(q, s + \delta s)$ construct a magnet of strength $|\underline{m}| = \frac{\Gamma}{2} \int_{\partial \mathcal{R}} \underline{x} \times d\underline{s}$, centered at the center of \mathcal{R} , and oriented so that the sum of the rectangles adds up to the original vortex loop C . Note that for large-scale organized vortex loops, the magnetization representation is less efficient than the standard vortex representation.

To go from a magnetization to a vortex representation, replace each magnet by a vortex loop, reversing the last part of the procedure of the preceding paragraph. To construct macroscopic vortices, if indeed they exist, go through the cut-and-paste procedures described e.g. in [7]. A natural question is, how large are the vortex loops that can be constructed from a given collection of magnets? This turns out to be a problem in percolation theory, studied in [10] (see also [21]). The answer depends on the temperature of the magnet/vortex system; in a fully turbulent regime, $|T| = \infty$, there exist vortex structures of arbitrarily large size, limited only by the physical extent of the turbulence.

Since $Z = \sum |\underline{m}^{(k)}|$ is monotonically increasing, a thermal equilibrium cannot be expected in a magnet system, unless appropriate assumptions are added. In analogy with the properties of

vortex systems, one can stop the growth of Z by fiat, or one can construct a statistically stationary inertial state by steadily creating well-separated magnets and removing nearby magnets with nearly opposing orientation; that steady state can then be approximated by a Gibbs equilibrium. In either case, some of the advantages of the Hamiltonian structure are lost. As in the equivalent vortex representation, one finds positive and negative temperature states, separated by the attracting $|T| = \infty$ maximum entropy state.

For $T > 0$ the most likely states are the ones that minimize the Hamiltonian H , and for $T < 0$ the most likely states are the ones that maximize it.

Renormalization in magnetization variables. The $|T| = \infty$ turbulent state is a critical state that shares many properties with the critical order/disorder superfluid transition state at low positive temperatures, as described by the Kosterlitz-Thouless and Shenoy-Williams theories [12],[17],[30],[31]. Indeed, in a simplified “ $2\frac{1}{2}$ ” dimensional system one can draw a curve that connects these states and along which certain universality properties hold [10]; it is therefore reasonable to hope that the renormalization analysis near the superfluid transition can shed light in the turbulent state and provide an insight into ways of renormalizing, i.e., simplifying, calculations. In a renormalization analysis, one persistently removes small scales from a calculation in such a way that the equilibrium, or the dynamics, are unchanged on a macroscopic scale. When $T > 0$, a large magnet, or vortex loop, polarizes smaller ones, i.e., at or near equilibrium, the greater likelihood of lower energies makes it likely that the smaller loops are arranged so as to reduce the energy. The removal of small scale structures requires a decrease in vortex strength to make up for it. For $T < 0$, the opposite is true: near equilibrium a large magnet anti-polarizes the medium, and renormalization requires a strengthening of vortex lines. On the $|T| = \infty$ boundary between positive and negative T , one should be able to remove small loops with impunity provided one does not otherwise perturb the system. This removal of small scales is the idea behind the “hairpin re-

moval” of refs. [4],[7]. Analogous constructions have been proposed by Sethian [29]. Note that such “hairpin removal”, which is entirely equivalent to magnet merger, is much more easily performed in a magnet representation, where the topological constraints are trivial and conservation of impulse can be imposed exactly.

Conclusions. We have developed the magnetization representation [1]. We have shown that the change of variables between vortex and magnet representation is easy and can be made locally, i.e., hybrid vortex/magnet representations are readily implemented. It is natural to represent large scales by means of vortices and small scales by means of magnets, both for renormalization purposes and for the approximation of diffusion. For a detailed discussion of such hybrids, see [2]. Renormalization can be carried out locally in a magnetization representation, as seems to be desirable in view of the inhomogeneity of the vortex temperature field.

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